OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS 4726

Further Pure Mathematics 2

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Monday 10 January 2011
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a scientific or graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

© OCR 2011 [A/102/2699]
RP–0E25
OCR is an exempt Charity

1 Use the substitution $t = \tan \frac{1}{2}x$ to find $\displaystyle\int \frac{1}{1 + \sin x + \cos x}\,dx$. **[5]**

2 It is given that $f(x) = \tanh^{-1} x$.

(i) Show that $f'''(x) = \dfrac{2(1 + 3x^2)}{(1 - x^2)^3}$. **[5]**

(ii) Hence find the Maclaurin series for $f(x)$, up to and including the term in x^3. **[3]**

3 The function f is defined by $f(x) = \dfrac{5ax}{x^2 + a^2}$, for $x \in \mathbb{R}$ and $a > 0$.

(i) For the curve with equation $y = f(x)$,

(a) write down the equation of the asymptote, **[1]**

(b) find the range of values that y can take. **[4]**

(ii) For the curve with equation $y^2 = f(x)$, write down

(a) the equation of the line of symmetry, **[1]**

(b) the maximum and minimum values of y, **[2]**

(c) the set of values of x for which the curve is defined. **[1]**

4 **(i)** Use the definitions of hyperbolic functions in terms of exponentials to prove that

$$8 \sinh^4 x \equiv \cosh 4x - 4\cosh 2x + 3.$$ **[4]**

(ii) Solve the equation

$$\cosh 4x - 3\cosh 2x + 1 = 0,$$

giving your answer(s) in logarithmic form. **[5]**

5 The equation

$$x^3 - 5x + 3 = 0 \qquad \text{(A)}$$

may be solved by the Newton-Raphson method. Successive approximations to a root are denoted by $x_1, x_2, \ldots, x_n, \ldots$.

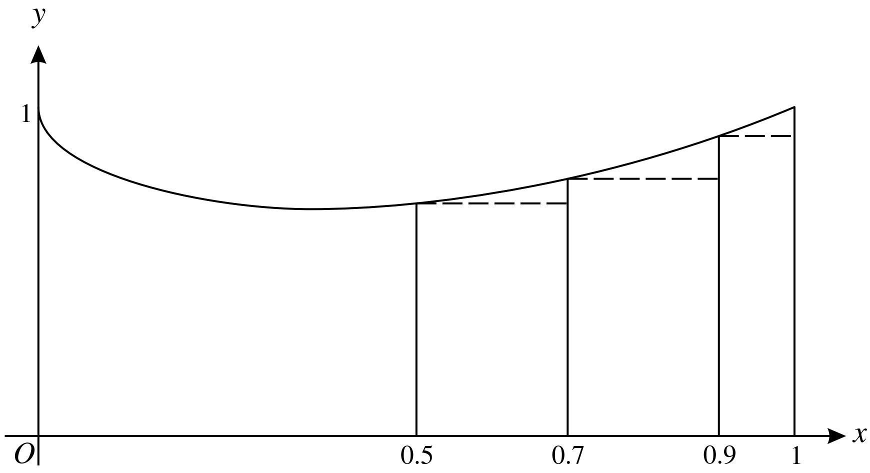
(i) Show that the Newton-Raphson formula can be written in the form $x_{n+1} = F(x_n)$, where

$$F(x) = \frac{2x^3 - 3}{3x^2 - 5}.$$ **[3]**

(ii) Find $F'(x)$ and hence verify that $F'(\alpha) = 0$, where α is any one of the roots of equation (A). **[3]**

(iii) Use the Newton-Raphson method to find the root of equation (A) which is close to 2. Write down sufficient approximations to find the root correct to 4 decimal places. **[3]**

6



The diagram shows the curve $y = \mathrm{f}(x)$, defined by

$$\mathrm{f}(x) = \begin{cases} x^x & \text{for } 0 < x \leqslant 1, \\ 1 & \text{for } x = 0. \end{cases}$$

(i) By first taking logarithms, show that the curve has a stationary point at $x = \mathrm{e}^{-1}$. **[3]**

The area under the curve from $x = 0.5$ to $x = 1$ is denoted by A.

(ii) By considering the set of three rectangles shown in the diagram, show that a lower bound for A is 0.388. **[2]**

(iii) By considering another set of three rectangles, find an upper bound for A, giving 3 decimal places in your answer. **[2]**

The area under the curve from $x = 0$ to $x = 0.5$ is denoted by B.

(iv) Draw a diagram to show rectangles which could be used to find lower and upper bounds for B, using not more than three rectangles for each bound. (You are not required to find the bounds.) **[3]**

7 A curve has polar equation $r = 1 + \cos 3\theta$, for $-\pi < \theta \leqslant \pi$.

(i) Show that the line $\theta = 0$ is a line of symmetry. **[2]**

(ii) Find the equations of the tangents at the pole. **[3]**

(iii) Find the exact value of the area of the region enclosed by the curve between $\theta = -\frac{1}{3}\pi$ and $\theta = \frac{1}{3}\pi$. **[5]**

8 **(i)** Without using a calculator, show that $\sinh(\cosh^{-1} 2) = \sqrt{3}$. **[2]**

(ii) It is given that, for non-negative integers n,

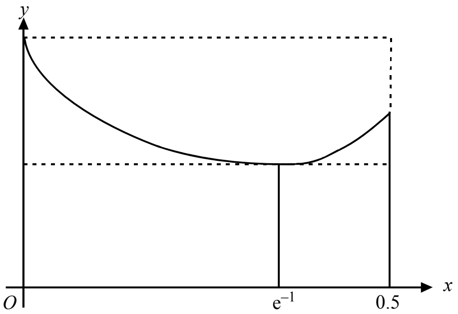
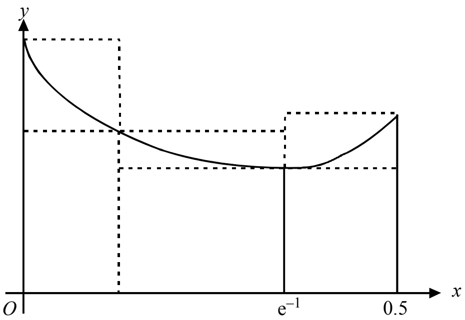
$$I_n = \int_0^{\beta} \cosh^n x \,\mathrm{d}x, \quad \text{where } \beta = \cosh^{-1} 2.$$

Show that $nI_n = 2^{n-1}\sqrt{3} + (n-1)I_{n-2}$, for $n \geqslant 2$. **[6]**

(iii) Evaluate I_5, giving your answer in the form $k\sqrt{3}$. **[4]**

1	$t = \tan\frac{1}{2}x \Rightarrow \mathrm{d}t = \frac{1}{2}\sec^2\frac{1}{2}x\,\mathrm{d}x = \frac{1}{2}(1+t^2)\,\mathrm{d}x$	B1		For correct result **AEF** (may be implied)
	$\int \frac{1}{1+\sin x+\cos x}\,\mathrm{d}x = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,\mathrm{d}t$	M1 A1		For substituting throughout for x For correct unsimplified t integral
	$= \int \frac{1}{1+t}\,\mathrm{d}t = \ln\lvert 1+t\rvert\ (+c)$	M1		For integrating (even incorrectly) to $a\ln\lvert \mathrm{f}(t)\rvert$. Allow $\lvert\ \rvert$ or $(\ )$
	$= \ln k\left\lvert 1+\tan\frac{1}{2}x\right\rvert\ (+c)$	A1	**5**	For correct x expression k may be numerical, c is not required
			$\boxed{5}$	
2 (i)	$\mathrm{f}(x) = \tanh^{-1}x,\ \mathrm{f}'(x) = \frac{1}{1-x^2},\ \mathrm{f}''(x) = \frac{2x}{(1-x^2)^2}$	M1		For quoting $\mathrm{f}'(x) = \frac{1}{1\pm x^2}$ and attempting to differentiate $\mathrm{f}'(x)$
		A1		For $\mathrm{f}''(x)$ correct **WWW**
	$\mathrm{f}'''(x) =$ $\frac{2(1-x^2)^2 - 2x.2(1-x^2).-2x}{(1-x^2)^4}$ *OR* $\frac{2x.4x}{(1-x^2)^3} + \frac{2}{(1-x^2)^2}$	M1 A1		For using quotient *OR* product rule on $\mathrm{f}''(x)$ For correct unsimplified $\mathrm{f}'''(x)$
	$= \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$ *OR* $\frac{8x^2}{(1-x^2)^3} + \frac{2(1-x^2)}{(1-x^2)^3}$			
	$= \frac{2(1+3x^2)}{(1-x^2)^3}$	A1	**5**	For simplified $\mathrm{f}'''(x)$ **WWW AG**
(ii)	$\mathrm{f}(0) = 0,\ \mathrm{f}'(0) = 1,\ \mathrm{f}''(0) = 0$	B1√		For all values correct (may be implied) f.t. from **(i)**
	$\mathrm{f}'''(0) = 2 \Rightarrow \tanh^{-1}x = x + \frac{1}{3}x^3$	M1 A1	**3**	For evaluating $\mathrm{f}'''(0)$ and using Maclaurin expansion For correct series
			$\boxed{8}$	
3 (i)(a)	Asymptote $y = 0$	B1	**1**	For correct equation (allow x-axis)
(b)	METHOD 1 $y = \frac{5ax}{x^2+a^2} \Rightarrow yx^2 - 5ax + a^2y = 0$	M1 M1		For expressing as a quadratic in x For using $b^2 - 4ac \lesseqgtr 0$
	$b^2 \geqslant 4ac \Rightarrow 25a^2 \geqslant 4a^2y^2 \Rightarrow -\frac{5}{2} \leqslant y \leqslant \frac{5}{2}$	A1 A1	**4**	For $25a^2 - 4a^2y^2$ seen or implied For correct range
	METHOD 2 $y = \frac{5ax}{x^2+a^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-5a\left(x^2-a^2\right)}{\left(x^2+a^2\right)^2}$	M1*		For differentiating y by quotient *OR* product rule
	$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = \pm a \Rightarrow y = \pm\frac{5}{2}$	A1 M1		For correct values of x For finding y values and giving argument for range
	Asymptote, sketch etc $\Rightarrow -\frac{5}{2} \leqslant y \leqslant \frac{5}{2}$	A1 (*dep)		For correct range
(ii)(a)	$y = 0$	B1	**1**	For correct equation (allow x-axis)
(b)	Maximum $\sqrt{\frac{5}{2}}$, minimum $-\sqrt{\frac{5}{2}}$	B1√ B1√	**2**	For correct maximum f.t. from **(i)(b)** For correct minimum f.t. from **(i)(b)** Allow decimals
(c)	$x \geqslant 0$	B1	**1**	For correct set of values (allow in words)
			$\boxed{9}$	

4 (i)	$8\sinh^4 x \equiv \frac{8}{16}\left(e^x - e^{-x}\right)^4$	B1		$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ seen or implied
	$\equiv \frac{8}{16}\left(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}\right)$	M1		For attempt to expand $(\dots)^4$ by binomial theorem *OR* otherwise
	$\equiv \frac{1}{2}\left(e^{4x} + e^{-4x}\right) - \frac{4}{2}\left(e^{2x} + e^{-2x}\right) + \frac{6}{2}$	M1		For grouping terms for $\cosh 4x$ *or* $\cosh 2x$ *OR* using e^{4x} *or* e^{2x} expressions from RHS
	$\equiv \cosh 4x - 4\cosh 2x + 3$	A1	4	For correct expression **AG**
	SR may be done wholly from RHS to LHS	M1 M1 B1 A1		Evidence of factorising required for 2nd M1
(ii)	METHOD 1 $\cosh 4x - 3\cosh 2x + 1 = 0$			
	$\Rightarrow (8\sinh^4 x + 4\cosh 2x - 3) - 3\cosh 2x + 1 = 0$	M1		For using **(i)** and $\cosh 2x \equiv \pm 1 \pm 2\sinh^2 x$
	$\Rightarrow 8\sinh^4 x + 2\sinh^2 x - 1 = 0$	A1		For correct equation
	$\Rightarrow \left(4\sinh^2 x - 1\right)\left(2\sinh^2 x + 1\right) = 0 \Rightarrow \sinh x = \pm\frac{1}{2}$	M1 A1		For solving their quartic for $\sinh x$ For correct $\sinh x$ (ignore other roots)
	$\Rightarrow x = \ln\left(\pm\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) = \pm\ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$	A1√	5	For correct answers (and no more) f.t. from their value(s) for $\sinh x$
	SR Similar scheme for $8\cosh^4 x - 14\cosh^2 x + 5 = 0 \Rightarrow \cosh x = \frac{1}{2}\sqrt{5} \Rightarrow x = \pm\ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$			
	METHOD 2 $\cosh 4x - 3\cosh 2x + 1 = 0$			
	$\Rightarrow (2\cosh^2 2x - 1) - 3\cosh 2x + 1 = 0$	M1		For using $\cosh 4x \equiv \pm 2\cosh^2 2x \pm 1$
	$\Rightarrow 2\cosh^2 2x - 3\cosh 2x = 0$	A1		For correct equation
	$\Rightarrow \cosh 2x = \frac{3}{2} \Rightarrow x = \frac{1}{2}\ln\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right)$	M1 A1		For solving for $\cosh 2x$ For correct $\cosh 2x$ (ignore others)
	$= \pm\frac{1}{2}\ln\left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right)$	A1√		For correct answers (and no more) f.t. from value(s) for $\cosh 2x$
	METHOD 3 Put all into exponentials	M1		For changing to $e^{\pm kx}$
	$\Rightarrow e^{4x} - 3e^{2x} + 2 - 3e^{-2x} + e^{-4x} = 0$	A1		For correct equation
	$\Rightarrow \left(e^{4x} + 1\right)\left(e^{4x} - 3e^{2x} + 1\right) = 0$	M1		For solving for e^{2x}
		A1		For correct e^{2x} (ignore others)
	$\Rightarrow e^{2x} = \frac{1}{2}\left(3 \pm \sqrt{5}\right) \Rightarrow x = \frac{1}{2}\ln\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right)$	A1√		For correct answers (and no more) f.t. from value(s) for e^{2x}
			$\boxed{9}$	
5 (i)	$x_{n+1} = x_n - \dfrac{x_n^3 - 5x_n + 3}{3x_n^2 - 5} = \dfrac{2x_n^3 - 3}{3x_n^2 - 5}$	M1 A1 A1	3	For attempt at N-R formula For correct N-R expression For correct answer (necessary details needed) **AG** Allow omission of suffixes
(ii)	$F'(x) =$	M1		For using quotient *OR* product rule to find $F'(x)$
	$\dfrac{6x^2\left(3x^2 - 5\right) - 6x\left(2x^3 - 3\right)}{\left(3x^2 - 5\right)^2} = \dfrac{6x\left(x^3 - 5x + 3\right)}{\left(3x^2 - 5\right)^2}$	M1		For factorising numerator to show $k\left(x^3 - 5x + 3\right)$
	$F'(\alpha) = \dfrac{6\alpha\left(\alpha^3 - 5\alpha + 3\right)}{\left(3\alpha^2 - 5\right)^2} = 0$ since $\alpha^3 - 5\alpha + 3 = 0$	A1	3	For correct explanation of **AG**
(iii)	$x_1 = 2 \Rightarrow 1.85714,\ 1.83479,\ 1.83424,\ 1.83424$	B1		First iterate correct to at least 4 d.p. *OR* $\frac{13}{7}$
	$(\alpha =)\ 1.8342$	B1 B1	3	For 2 equal iterates to at least 4 d.p. For correct α to 4 d.p.
	SR For starting value leading to another root allow up to B1 B1 B0			Allow answer rounding to 1.8342 **SR** If not N-R, B0 B0 B0
			$\boxed{9}$	

6 (i)	$y=x^x \Rightarrow \ln y = x\ln x \Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=1+\ln x$	M1		For differentiating $\ln y$ *OR* $x\ln x$ w.r.t. x
	$\frac{\mathrm{d}y}{\mathrm{d}x}=x^x\left(1+\ln x\right)=0 \Rightarrow \ln x=-1 \Rightarrow x=\mathrm{e}^{-1}$	A1		For $\left(1+\ln x\right)$ seen or implied
		A1	3	For correct x-value from fully correct working **AG**
(ii)	$A>0.2\times0.5^{0.5}+0.2\times0.7^{0.7}+0.1\times0.9^{0.9}$	M1		For areas of 3 lower rectangles
	$\Rightarrow A>0.3881(858)>0.388$	A1	2	For lower bound rounding to **AG**
(iii)	$A<0.2\times0.7^{0.7}+0.2\times0.9^{0.9}+0.1\times1^1$	M1		For areas of 3 upper rectangles
	$\Rightarrow A<0.4377(177)<0.438$	A1	2	For upper bound rounding to 0.438
(iv)		M1		Consider rectangle of height $\mathrm{f}(\mathrm{e}^{-1})$
		A1		Use at least 1 lower rectangle, height $\mathrm{f}(\mathrm{e}^{-1})$
		B1	3	Use at least 1 upper rectangle, height $\mathrm{f}(0)$
				**SR** If more than one rectangle is used for either bound, they must be shown correctly
		$\boxed{10}$		
7 (i)	$\cos 3\theta=\cos(-3\theta)$ *OR* $\cos\theta=\cos(-\theta)$ for all θ	M1		For a correct procedure for symmetry related to the equation *OR* to $\cos 3\theta$
	$\Rightarrow$ equation is unchanged, so symmetrical about $\theta=0$	A1	2	For correct explanation relating to equation **AG**
(ii)	$r=0 \Rightarrow \cos 3\theta=-1$	M1		For obtaining equation for tangents
	$\Rightarrow \theta=\pm\frac{1}{3}\pi, \pi$	A1		A1 for any 2 values
		A1	3	A1 for all, no extras (ignore outside range)
(iii)	$\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}\frac{1}{2}\left(1+\cos 3\theta\right)^2\,(\mathrm{d}\theta)$	B1		For correct integral with limits soi (limits may be $\left[0, \frac{1}{3}\pi\right]$ at any stage)
	$=\frac{1}{2}\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}1+2\cos 3\theta+\cos^2 3\theta\,\mathrm{d}\theta$	M1*		For multiplying out, giving at least 2 terms
	$=\frac{1}{2}\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}1+2\cos 3\theta+\frac{1}{2}\left(1+\cos 6\theta\right)\mathrm{d}\theta$	M1		For integration to $A\theta+B\sin 3\theta+C\sin 6\theta$ **AEF**
	$=\frac{1}{2}\left[\theta+\frac{2}{3}\sin 3\theta+\left(\frac{1}{2}\theta+\frac{1}{12}\sin 6\theta\right)\right]_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}$	M1 (*dep)		For completing integration and substituting their limits into terms in $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n\theta$
	$=\frac{1}{2}\pi$	A1	5	For correct area **WWW**
		$\boxed{10}$		

8	**(i)**	METHOD 1 $\sinh(\cosh^{-1}2) =$ $\sinh\beta = \sqrt{\cosh^2\beta - 1} = \sqrt{2^2 - 1} = \sqrt{3}$	M1 A1 **2**	For appropriate use of $\sinh^2\theta = \cosh^2\theta - 1$ For correct verification to **AG**
		METHOD 2 $\sinh^{-1}\sqrt{3} = \ln\left(\sqrt{3}+2\right)$, $\cosh^{-1}2 = \ln\left(2+\sqrt{3}\right)$ $\Rightarrow \sinh(\cosh^{-1}2) = \sqrt{3}$	M1 A1	For attempted use of ln forms of $\sinh^{-1}x$ and $\cosh^{-1}x$ For both ln expressions seen
		METHOD 3 $\cosh^{-1}2 = \ln\left(2+\sqrt{3}\right)$ $\sinh\left(\cosh^{-1}2\right) = \frac{1}{2}\left(e^{\ln\left(2+\sqrt{3}\right)} - e^{-\ln\left(2+\sqrt{3}\right)}\right)$ $= \frac{1}{2}\left(2+\sqrt{3} - \left(2-\sqrt{3}\right)\right) = \sqrt{3}$	M1 A1	For use of ln form of $\cosh^{-1}x$ and definition of $\sinh x$ For correct verification to **AG** **SR** Other similar methods may be used Note that $\ln\left(2+\sqrt{3}\right) = -\ln\left(2-\sqrt{3}\right)$
	(ii)	$I_n = \int_0^\beta \cosh^n x\,dx$ $= \left[\sinh x.\cosh^{n-1}x\right]_0^\beta - \int_0^\beta \sinh^2 x.(n-1)\cosh^{n-2}x\,dx$ $= \sinh\beta.\cosh^{n-1}\beta - (n-1)\int_0^\beta \left(\cosh^2 x - 1\right)\cosh^{n-2}x\,dx$ $= 2^{n-1}\sqrt{3} - (n-1)\left(I_n - I_{n-2}\right)$ $\Rightarrow nI_n = 2^{n-1}\sqrt{3} + (n-1)I_{n-2}$	M1* A1 M1 (*dep) A1 B1 A1 **6**	For attempt to integrate $\cosh x.\cosh^{n-1}x$ by parts For correct first stage of integration (ignore limits) For using $\sinh^2 x = \cosh^2 x - 1$ For $(n-1)\left(I_n - I_{n-2}\right)$ correct For $2^{n-1}\sqrt{3}$ correct at any stage For correct result **AG**
	(iii)	$I_1 = \int_0^\beta \cosh x\,dx = \sinh\beta = \sqrt{3}$ $I_3 = \frac{1}{3}\left(2^2\sqrt{3} + 2\sqrt{3}\right) = 2\sqrt{3}$ $I_5 = \frac{1}{5}\left(2^4\sqrt{3} + 8\sqrt{3}\right) = \frac{24}{5}\sqrt{3}$	B1 M1 A1 A1 **4**	For correct value For using **(ii)** with $n=3$ *OR* $n=5$ For $I_3 = \frac{1}{3}\left(2^2\sqrt{3} + 2I_1\right)$ *OR* $I_5 = \frac{1}{5}\left(2^4\sqrt{3} + 4I_3\right)$ For correct value
			$\boxed{12}$	